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TRIGONOMETRY

GRIFFIN.

AN

ELEMENTARY TREATISE

ON

PLANE TRIGONOMETRY,

DESIGNED CHIEFLY FOR

THE USE OF CANDIDATES PREPARING FOR THE ORDINARY
TERM EXAMINATIONS IN THE UNIVERSITIES.

BY

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PREFACE.

PLANE TRIGONOMETRY was originally restricted to the investigation of the relations between the Elements of Plane Triangles. It is now extended to the investigation of the relations of Arcs and Angles in general, and to the properties which connect the sides and angles of any plane rectilineal figure.

The invention of Logarithms by Napier was a valuable aid to the practical application of Trigonometry, and it is now impossible to treat of the latter without devoting at least a short chapter to the former.

Many of the existing Treatises on Trigonometry are wanting in connexion, and also omit to point out to the Student that there is a definite number of elementary relations connecting the Trigonometrical Ratios, and fail to show in a systematic manner how they may be discovered.

These remarks are made not with a design to criticize the works of others, but as an indirect apology for the production of the present Treatise.

I have given the proof of the four fundamental formulæ as propounded to my pupils during the past twenty-five years, and I cannot avoid expressing my disapproval of the mutilation of the proof by writers who have adopted it in part.

ROBERT W. GRIFFIN.

19, Trinity College, Dublin,
February, 1883.

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PLANE TRIGONOMETRY.

CHAPTER I.

1. **Division of the Circle.**—If the circumference of any circle be divided into 360 equal parts, the angle subtended at the centre by any one of these parts is called a *degree.* The sixtieth part of a degree is called a *minute*, and the sixtieth part of a minute is called a *second.* The symbols °, ′, ″, are used as abbreviations of the words *degrees*, *minutes*, *seconds.* Thus 57° 17′ 45″ denote 57 degrees 17 minutes 45 seconds.

2. **Method of comparing Angles.**—In the same or in equal circles the angles at the centre are proportional to the arcs on which they stand (Eu. 33 VI.), and therefore angles are frequently compared by comparing the arcs which subtend them.

3. **Angular Unit.**—The angle subtended at the centre of a circle by an arc of the circumference, equal in length to the radius (sometimes called a *radian*), is taken as the *angular unit;* and the ratio of any other angle to the angular unit, *or* the ratio of the arc which subtends the angle to the radius, is called the *circular measure* of the angle.

4. **Ratio of Circumference to Diameter.**—The ratio of the circumference of a circle to its diameter, what-

ever be the diameter, is constant. The student must here take for granted that the numerical value of this ratio is nearly 3·14159, which for convenience we shall denote by π.

If C denote the circumference of a circle whose radius $= r$, then

$$\frac{C}{\text{diameter}} = \frac{C}{2r} = \pi;$$

$$\therefore\ C = 2\pi r.$$

5. **Number of Degrees in Angular Unit.**—If an arc of the circumference be supposed cut off equal to the radius, then

$$C : r = 360^\circ : \text{number of degrees in angular unit};$$

(Euc. 33, VI.)

$$\begin{aligned}\therefore\ \text{number of degrees in angular unit} &= \frac{360^\circ r}{C}\\ &= \frac{360^\circ r}{2\pi r}\\ &= \frac{180^\circ}{\pi}\\ &= \frac{180^\circ}{3{\cdot}14159}\\ &= 57^\circ{\cdot}29578 \text{ nearly.}\end{aligned}$$

Cor.—Number of seconds in the angular unit $= 206265$.

6. **Circular Measure of a Right Angle.**—The circular measure of $360^\circ = \dfrac{\text{circumference}}{\text{radius}} = \dfrac{2\pi r}{r} = 2\pi$; therefore the circular measure of $90^\circ = \dfrac{\pi}{2}$.

Cor.—Circular measure of $n^\circ = \dfrac{n\pi}{180}$.

7. Relation between the Length of Arc and the Number of Seconds in the Corresponding Angle.—If A'' be the number of seconds in any angle, a the length of the corresponding arc, and r the radius of the circle; then

$$\frac{a}{r} = \frac{A''}{206265''}. \qquad \text{(Euc. 33, VI.)}$$

Cor.—Hence the circular measure of A'' is also

$$= \frac{A''}{206265''}.$$

Examples.

1. How many seconds in the angle whose circular measure $= \cdot 2$?

$$\frac{A''}{206265''} = \cdot 2; \quad \therefore \ A'' = 41253''.$$

2. What angle will a man 6 feet high subtend at a point in the same horizontal plane with the man 1 mile distant?

The 6 feet may be considered as an arc of a circle of 1 mile (5280 feet) radius;

$$\therefore \ \frac{6}{5280} = \text{circular measure of } \angle = \frac{A''}{206265''};$$

$$\therefore \ A'' = \frac{6 \times 206265''}{5280} = 234''.$$

3. A tower 120 feet high, standing on a horizontal plane, subtends an angle of $5'$ at the eye of an observer: calculate the distance of the tower.

$$\frac{120}{r} = \frac{300}{206265},$$

$$\therefore \ r = \frac{120 \times 206265}{300} = 82506 \text{ feet.}$$

CHAPTER II.

8. **Method of Computing Angles.**—In Plane Trigonometry the computation of angles is for the most part carried on by means of the ratios of right lines drawn with regard to the angle, which ratios are not measures of the angles, but from which the angles may be computed. These ratios are called *sine, cosine, tangent, cotangent, secant, cosecant, versed sine, coversed sine;* but for shortness written *sin, cos, tan, cot, sec, cosec, versin, covers.*

9. **Definitions of the Trigonometrical Ratios.**—From any point on one leg of an angle let fall a perpendicular on the other leg.

If the angle be denoted by A, the distance of the point from the vertex of the angle by c, the perpendicular by a,

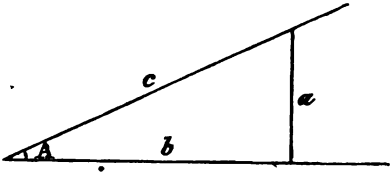

Fig. 1.

and the part intercepted on the other leg between the foot of the perpendicular and the vertex of the angle by b; then we shall define the above ratios as follows :—

$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \tan A = \frac{a}{b},$$

$$\operatorname{cosec} A = \frac{c}{a}, \text{ or the reciprocal of } \sin A,$$

$$\sec A = \frac{c}{b}, \quad \text{,,} \quad \text{,,} \quad \cos A,$$

$$\cot A = \frac{b}{a}, \quad \text{,,} \quad \text{,,} \quad \tan A,$$

$$\operatorname{versin} A = 1 - \cos A, \quad \operatorname{coversin} A = 1 - \sin A.$$

(NOTE.—The reciprocal of any quantity is unity divided by that quantity, and therefore the reciprocal of a fraction is that fraction inverted.)

10. **Inverse Trigonometrical Functions.**—The following notation is also used, viz.:

$$\sin^{-1} x, \quad \cos^{-1} y, \quad \tan^{-1} z, \quad \&c.,$$

to denote respectively the angle whose sine is x, the angle whose cosine is y, the angle whose tangent is z.

Hence, if $x = \sin A$, then $A = \sin^{-1} x$,

$y = \cos B$, ,, $B = \cos^{-1} y$,

$z = \tan C$, ,, $C = \tan^{-1} z$.

Also, $\sin(\sin^{-1} x) = x$, $\cos(\cos^{-1} y) = y$, $\tan(\tan^{-1} z) = z$.

Examples.

1. Construct the angle whose tangent $= \frac{3}{4}$, and find all the other trigonometrical ratios.

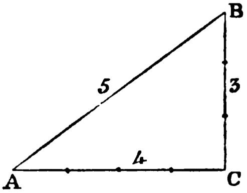

Fig. 2.

Lay down a line BC equal to 3 from a scale of equal parts. Erect CA

perpendicular to BC, which make equal to 4 from the same scale. Join BC. Then BAC is the required angle.

For $$\tan \text{ of } \angle BAC = \frac{BC}{CA} = \frac{3}{4};$$

$$\therefore \angle BAC = \tan^{-1}\frac{3}{4}.$$

Also $$AB^2 = AC^2 + CB^2 = 4^2 + 3^2 = 25;$$

$$\therefore AB = 5.$$

Hence $$\sin A = \frac{3}{5}, \qquad \operatorname{cosec} A = \frac{5}{3},$$

$$\cos A = \frac{4}{5}, \qquad \sec A = \frac{5}{4},$$

$$\tan A = \frac{3}{4}, \qquad \cot A = \frac{4}{3},$$

$$\operatorname{versin} A = \frac{1}{5}, \qquad \operatorname{coversin} A = \frac{2}{5}.$$

2. Construct the angle whose cosine $= \frac{6}{7}$, and find all the other trigonometrical ratios.

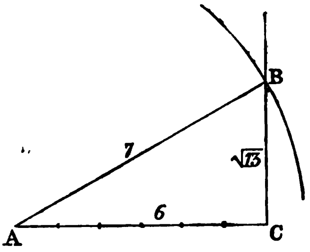

Fig. 3.

Lay down a line AC equal to 6 from a scale of equal parts. At C erect a perpendicular to AC of an indefinite length. With the centre A, and radius equal to 7 from the same scale, describe a circle cutting the perpendicular in B. Join AB. Then the angle BAC is the required angle.

For $$\cos \text{ of } \angle BAC = \frac{AC}{AB} = \frac{6}{7};$$

$$\therefore \angle BAC = \cos^{-1}\frac{6}{7}.$$

Also
$$AB^2 = AC^2 + CB^2,$$
$$49 = 6^2 + CB^2;$$
$$\therefore\ CB^2 = 49 - 36;$$
$$\therefore\ CB = \sqrt{13}.$$

Hence
$$\sin A = \frac{\sqrt{13}}{7}, \qquad \operatorname{cosec} A = \frac{7}{\sqrt{13}},$$
$$\cos A = \frac{6}{7}, \qquad \sec A = \frac{7}{6},$$
$$\tan A = \frac{\sqrt{13}}{6}, \qquad \cot A = \frac{6}{\sqrt{13}},$$
$$\operatorname{versin} A = 1 - \frac{6}{7} = \frac{1}{7}, \qquad \operatorname{coversin} A = 1 - \frac{\sqrt{13}}{7}.$$

3. Construct the angle whose sine $= \frac{1}{\sqrt{3}}$, and find all the other ratios.

4. Construct the angle whose cosec $= 5$, and find, &c.

5. Construct the angle whose cot $= \cdot 7$, and find, &c.

6. Construct the angle whose versin $= \cdot 4$, and find, &c.

7. Show that $\tan\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{4}$.

11. Expressions for the Trigonometrical Ratios in Terms of Tangent.—Let $\tan A = a$, then $A = \tan^{-1} a$. If a right-angled triangle be constructed, of which the sides about the right angle are a and 1, then the tangent of the angle opposite the side $a = \frac{a}{1}$ or a;

therefore the angle opposite the side $a = \tan^{-1} a = A$.

Also the hypotenuse of $\triangle = \sqrt{1 + a^2}$;

$$\therefore\ \sin A = \frac{a}{\sqrt{1+a^2}} = \frac{\tan A}{\sqrt{1+\tan^2 A}},\quad \operatorname{cosec} A = \frac{\sqrt{1+\tan^2 A}}{\tan A},$$

$$\cos A = \frac{1}{\sqrt{1+a^2}} = \frac{1}{\sqrt{1+\tan^2 A}},\quad \sec A = \sqrt{1+\tan^2 A},$$

$$\cot A = \frac{1}{a} = \frac{1}{\tan A}.$$

12. **Expressions for the Trigonometrical Ratios in terms of Secant.**—Let $\sec A = a$; $\therefore\ A = \sec^{-1} a$. Conceive a right-angled triangle, of which the hypotenuse $= a$, and the base $= 1$; then the contained angle $= \sec^{-1} a = A$, and the remaining side $= \sqrt{a^2 - 1}$.

Hence

$$\sin A = \frac{\sqrt{a^2-1}}{a} = \frac{\sqrt{\sec^2 A - 1}}{\sec A},\quad \operatorname{cosec} A = \frac{\sec A}{\sqrt{\sec^2 A - 1}},$$

$$\cos A = \frac{1}{a} = \frac{1}{\sec A},$$

$$\tan A = \frac{\sqrt{a^2-1}}{1} = \sqrt{\sec^2 A - 1},\quad \cot A = \frac{1}{\sqrt{\sec^2 A - 1}}.$$

13. **Expressions for the Trigonometrical Ratios in Terms of Versine.**—Let $\operatorname{versin} A = a$; $\therefore \cos A = 1 - a$. Conceive a right-angled triangle, one side of which $= 1 - a$, and the hypotenuse $= 1$; then the cosine of the included angle $= 1 - a$, and the remaining side $= \sqrt{2a - a^2}$.

$$\text{Hence}\quad \sin A = \frac{\sqrt{2a - a^2}}{1} = \sqrt{2 \operatorname{versin} A - \operatorname{versin}^2 A},$$

$$\tan A = \frac{\sqrt{2a - a^2}}{1 - a} = \frac{\sqrt{2 \operatorname{versin} A - \operatorname{versin}^2 A}}{1 - \operatorname{versin} A},$$

&c., &c.

LE,

in Terms of all the others.

$$A = \sqrt{2\text{ versin }A - \text{versin}^2 A} = 1 - \text{covers }A.$$

$$\frac{\text{c}^2 A - 1}{\text{ec }A} = 1 - \text{versin }A = \sqrt{2\text{ covers }A - \text{covers}^2 A}.$$

$$\frac{1}{\text{c}^2 A - 1} = \frac{\sqrt{2\text{ versin }A - \text{versin}^2 A}}{1 - \text{versin }A} = \frac{1 - \text{covers }A}{\sqrt{2\text{ covers }A - \text{covers}^2 A}}.$$

$$A = \frac{1}{\sqrt{2\text{ versin }A - \text{versin}^2 A}} = \frac{1}{1 - \text{covers }A}.$$

$$\frac{\text{ec }A}{\text{c}^2 A - 1} = \frac{1}{1 - \text{versin }A} = \frac{1}{\sqrt{2\text{ covers }A - \text{covers}^2 A}}.$$

$$\text{c}^2 A - 1 = \frac{1 - \text{versin }A}{\sqrt{2\text{ versin }A - \text{versin}^2 A}} = \frac{\sqrt{2\text{ covers }A - \text{covers}^3 A}}{1 - \text{covers }A}.$$

$$\frac{\text{c}^2 A - 1}{\text{ec }A} = \text{versin }A = 1 - \sqrt{2\text{ covers }A - \text{covers}^2 A}.$$

$$\frac{A - 1}{\text{c }A} = 1 - \sqrt{2\text{ versin }A - \text{versin}^2 A} = \text{covers }A.$$

hat pointed out in Articles 11, 12, and 13.

14. **Definitions of Complement and Supplement.**—One *angle* is called the *complement* of another if the two angles added together make up a right angle.

One *arc* is called the *complement* of another if the two arcs added together make up a quadrant.

One *angle* is called the *supplement* of another if the two angles added together make up two right angles.

One *arc* is called the *supplement* of another if the two arcs added together make up the semicircumference.

In a triangle if $\angle C = 90°$, then $\angle A + \angle B = 90$;

$\therefore$ $\angle A$ is the complement of B, and *vice versâ.*

Denoting the sides opposite the angles A, B, C by a, b, c,

then $$\sin A = \frac{a}{c} = \cos B;$$

$\therefore$ *The sine of an angle is equal to the cosine of its complement.*

Also $$\tan A = \frac{a}{b} = \cot B;$$

$\therefore$ *The tangent of an angle is the cotangent of its complement.*

Again, $$\sec A = \frac{c}{a} = \operatorname{cosec} B;$$

$\therefore$ *The secant of an angle is the cosecant of its complement.*

It may assist the student in remembering the above, to point out that *co* is merely a contraction for complement; and that cosine is the complement sine, or sine of the complement; cotangent equal complement tangent, or the tangent of the complement, &c.

15. **Equivalent Expressions.**—(See fig. 1).

$$a^2 + b^2 = c^2;$$

$$\therefore\ \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2};$$

$$\therefore\ \sin^2 A + \cos^2 A = 1. \qquad (1)$$

Again,

$$a^2 + b^2 = c^2;$$

$$\therefore\ \frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2};$$

$$\therefore\ \tan^2 A + 1 = \sec^2 A. \qquad (2)$$

Similarly,

$$a^2 + b^2 = c^2;$$

$$\therefore\ \frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2};$$

$$\therefore\ 1 + \cot^2 A = \operatorname{cosec}^2 A. \qquad (3)$$

Also,

$$\frac{\sin a}{\cos a} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} = \tan A, \qquad (4)$$

and

$$\frac{\cos A}{\sin A} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a} = \cot A. \qquad (5)$$

The last five formulæ are so important that we have given an independent proof, though they are implied in the formulæ given in the table.

16. **Trigonometrical Functions of** 45°.—
Let $\angle AOB = 45°$: cut off $OM = 1$: draw $MP \perp OM$.

Since $\angle POM = 45°$, $\angle OPM = 45°$; $\therefore PM = OM = 1$,
and $OP = \sqrt{2}$; (Euc. 47, I.)

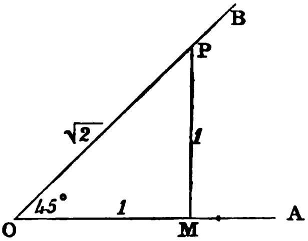

Fig. 4.

therefore $\sin 45° = \dfrac{MP}{PO} = \dfrac{1}{\sqrt{2}}$, $\operatorname{cosec} 45° = \sqrt{2}$,

$\cos 45° = \dfrac{OM}{OP} = \dfrac{1}{\sqrt{2}}$, $\sec 45° = \sqrt{2}$,

$\tan 45° = \dfrac{MP}{MO} = 1$, $\cot 45° = 1$.

17. **Trigonometrical Functions of** 60° **and** 30°.—
Let $\angle BOA = 60°$: cut off $OM = 1$: draw $MP \perp OM$.

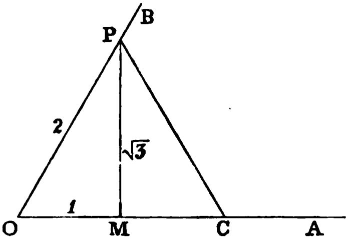

Fig. 6.

Make $\angle MPC = \angle MPO$. Then $\triangle OMP = CMP$.
(Euc. 26, I.)

Hence $\angle PCM = 60°$, and $\triangle OPC$ is equilateral; therefore $MC = OM = 1$, $OP = OC = 2$, and $PM = \sqrt{3}$.

(Euc. 48, I.)

$$\therefore\ \sin 60° = \cos 30° = \frac{\sqrt{3}}{2},\quad \operatorname{cosec} 60° = \sec 30° = \frac{2}{\sqrt{3}},$$

$$\cos 60° = \sin 30° = \frac{1}{2},\quad \sec 60° = \operatorname{cosec} 30° = 2,$$

$$\tan 60° = \cot 30° = \frac{\sqrt{3}}{1},\quad \cot 60° = \tan 30° = \frac{1}{\sqrt{3}}.$$

18. **Calculation of the Sines of Small Angles.**—Let a denote the arc which subtends the small angle A. From one extremity of the arc let fall a perpendicular (p) on the radius passing through the other extremity.

Now when the angle is very small the length of the arc a does not sensibly differ from the perpendicular p;

therefore $$\frac{p}{r} = \frac{a}{r} \text{ (very nearly).}$$

But $\frac{p}{r} = \sin A$, and $\frac{a}{r} =$ circular measure of A; therefore *The sine of a small angle is nearly equal to its circular measure.*

Examples.

1. Calculate the sin of 3 minutes.

$$\sin 3' = \frac{180}{206265} = \cdot 00087.$$

2. Show that $\sin 90° = 1$, $\cos 90° = 0$, $\tan 90° =$ infinity.

19. **Original Meaning of Sin, Cos, Tan, &c.**—It is right to inform the student that originally the terms *sin*, *cos*, *tan*, *sec*, &c., had reference to the arc of a circle.

Let the arc *BP* be denoted by a, and the corresponding angle at the centre by *A*:

PM = sin a = the perpendicular let fall from one extremity of the arc on the diameter passing through the other extremity of the arc.

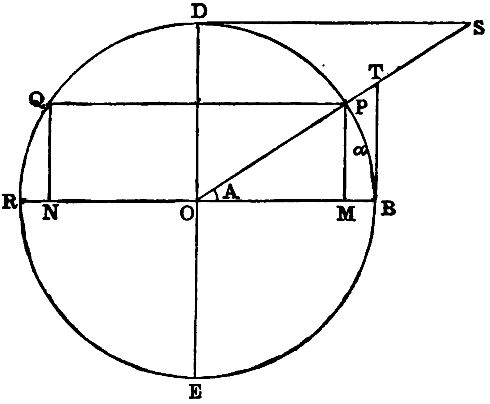

Fig. 6.

OM = cos a or the sine of the compliment = the distance of the foot of the sine from the centre.

BT = tan a = the part intercepted on a tangent drawn at one extremity of the arc, between the point of contact and a diameter passing through the other extremity.

OT = sec a = the part intercepted on a diameter passing through one extremity of the arc, between the centre and a tangent drawn at the other extremity of the arc.

DS = cot a = the tangent of the complement.

OS = cosec a = the secant of the complement.

MB = versin a

coversin a = versin of the complement.

The sin, cos, tan, &c., of the arc divided by the radius equal the sin, cos, tan, &c., of the corresponding angle.

Thus

$$PM = \sin a; \qquad \text{therefore} \quad \frac{PM}{OP} = \sin A,$$

$$OM = \cos a; \qquad \text{therefore} \quad \frac{OM}{OP} = \cos A,$$

$$BT = \tan a; \qquad \text{therefore} \quad \frac{BT}{OB} = \tan A.$$

20. **Convention with regard to the Algebraic Signs of the Trigonometrical Ratios.**—The algebraic signs of the different trigonometrical functions have also to be considered. If we regard those in the first quadrant as positive, and adopt as a criterion of *positive* and *negative*, that quantities change their sign in passing through *zero* or *infinity ;* observing also that sin 180° is zero, cos 90° is zero, cos 270° is zero; tan 90° is infinite, tan 180° is zero, and tan 270° is infinity; we shall find the following variations of sign:—

All arcs beginning at B and terminating at any point between B and R have *positive* sines; terminating between R and B have *negative* sines.

All arcs beginning at B and terminating at any point between B and D have *positive* cosines; terminating between D and E have *negative* cosines; and terminating between E and B have *positive* cosines.

All arcs beginning at B and terminating between B and D have *positive* tangents; terminating betwen D and R *negative;* terminating between R and E *positive,* and betwen E and B *negative.*

The cosecants, secants, and cotangents being the reciprocals of sines, cosines, and tangents, respectively, follow the same rules.

The algebraical signs of the trigonometrical functions are also determined in the same manner as in coordinate geometry. Thus the sine and tangent are positive or negative according as they are above or below the diameter *BR*, and the cosine and cotangent are positive or negative according as they are to the right or left of the diameter *DE*.

The following values may readily be deduced from the preceding:—

Angle.	Sin.	Cos.	Tan.	Cosec.	Sec.	Cot.	Vers.	Covers.
0	0	1	0	$\frac{1}{0}$	1	$\frac{1}{0}$	0	1
$\frac{\pi}{2}$	1	0	$\frac{1}{0}$	1	$\frac{1}{0}$	0	1	0
π	0	-1	0	$\frac{1}{0}$	-1	$-\frac{1}{0}$	2	1
$\frac{3}{2}\pi$	-1	0	$-\frac{1}{0}$	-1	$\frac{1}{0}$	0	1	2
2π	0	1	0	$\frac{1}{0}$	1	$\frac{1}{0}$	0	1

Note :— $\frac{1}{0}$ = infinity.

21. **Distinction between the Trigonometrical Functions of an Angle and those of an Arc.**— The student should note that the sine of an *arc* is a line, whilst the sine of an *angle* is a ratio. The same remark applies to the other trigonometrical functions.

22. **Relation connecting the Trigonometrical Function of an Arc and those of its Supplement.** See Fig. 6.

Through P draw $PQ \parallel BO$, and $QN \parallel DO$.

Now arc RQ = arc $BP = a$; $\therefore$ *arc* BQ *is the supplement of* a.

Also sin of arc $BQ = QN = PM = \sin a$;

$\therefore$ *the sine of an arc = the sine of its supplement.*

Again, cosine of an arc $BQ = ON$, and cosine $a = OM$;

$\therefore$ cosine of arc $BQ = -$ cosine a;

or, *The cosine of an arc = – cosine of its supplement.*

Hence *The tangent of an arc = – tangent of its supplement.*

Cor.—The sine of an ∠ *= the sine of its supplement.*

The cosine of an ∠ *= – the cosine of its supplement.*

The tan of an ∠ *= – the tangent of its supplement.*

CHAPTER III.

23. **Expression for the Sine of the Sum and the Cosine of the Difference of two Angles, in terms of the Sines and Cosines of the Angles themselves.**

Let the $\angle^s$ *EDF* and *EFD* of the $\triangle$ *DEF* be denoted respectively by *A* and *B*; the opposite sides *EF* and *ED* by *a* and *b*.

Draw *FM* $\perp$ *DE* produced, and *EN* $\perp$ *DF*.

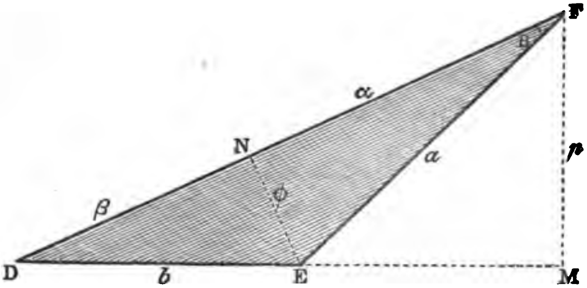

Fig. 7.

Let *FM*, *EN*, *NF*, *ND*, be denoted respectively by p, ϕ, α, β.

Now $\quad b \,.\, p = 2 \,.$ area of $\triangle\, DEF = \phi \,.\, (\alpha + \beta)$;

$$\therefore\ b \,.\, p = \phi \,.\, \alpha + \beta \,.\, \phi.$$

Divide by $b \,.\, a$ $$\frac{p}{a} = \frac{\phi}{b} \,.\, \frac{\alpha}{a} + \frac{\beta}{b} \,.\, \frac{\phi}{a}. \qquad (X)$$

But $$\angle FEM = A + B;$$

$$\therefore\ \sin(A + B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B.$$

If instead of denoting $\angle EDF$ by A, we make $\angle DFM = A$, then

$$\frac{p}{a} = \cos \text{ of } \angle EFM = \cos(A - B),$$

$$\frac{\phi}{b} = \sin \text{ of } \angle FDE \text{ or } \cos A,$$

and $$\frac{\beta}{b} = \cos \text{ of } \angle FDE \text{ or } \sin A.$$

Substituting in (X) we get

$$\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B.$$

24. Expression for the Sine of the Difference and the Cosine of the Sum of two Angles in terms of the Sines and Cosines of the Angles themselves.

Let the $\angle^s$ MDF and EFD be denoted respectively by A and B, and the sides EF and ED by a and b.

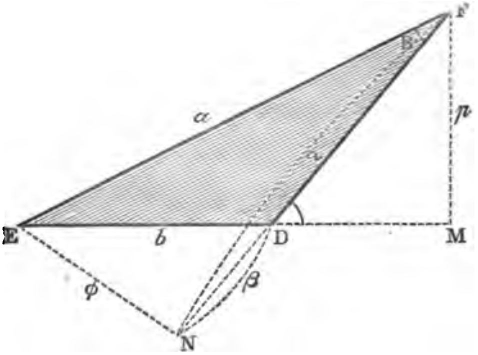

Fig. 8.

Draw $FM \perp ED$ produced, and $EN \perp FD$ produced.

Let FM, EN, NF, ND, be denoted respectively by p, ϕ, α, β

Now $b \,.\, p = 2$ area of $\triangle\, EDF = \phi \,.\, (a - \beta)$;

$$\therefore \quad bp = \phi \,.\, a - \beta \,.\, \phi.$$

Divide by $b \,.\, a$ $\quad \dfrac{p}{a} = \dfrac{\phi}{b} \,.\, \dfrac{a}{a} - \dfrac{\beta}{b} \,.\, \dfrac{\phi}{a}.$ $\quad (Y)$

But $\quad \angle\, FEM = A - B$;

$$\therefore \quad \sin(A - B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B. \qquad (3)$$

If, instead of denoting $\angle\, FDM$ by A, we make $\angle\, DFM = A$,

$$\frac{p}{a} = \cos \text{ of } \angle\, EFM = \cos(A + B),$$

$$\frac{\phi}{b} = \sin \text{ of } \angle\, NDE = \cos MFD \text{ or } \cos A,$$

$$\frac{\beta}{b} = \cos \angle\, NDE = \sin MFD \text{ or } \sin A.$$

Substituting in (Y) we get

$$\cos(A + B) = \cos A \cos B - \sin A \,.\, \sin B. \qquad (4)$$

The four formulæ proved in the last two Articles, viz.,

$$\sin(A + B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B,$$
$$\sin(A - B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B,$$
$$\cos(A + B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B,$$
$$\cos(A - B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B,$$

are called the *Fundamental Formulæ* of Trigonometry.

25. **Expressions for the Sine and Cosine of an Angle in terms of the Sine of twice the Angle.**

$$\sin(A + B) = \sin A \cos B + \cos A \,.\, \sin B.$$

Let $B = A$,

then $\quad \sin(A + B)$, or $\sin 2A = 2 \sin A \cos A$.

But $$1 = \sin^2 A + \cos^2 A;$$

$$\therefore\ 1 + \sin 2A = \sin^2 A + 2 \sin A \cos A + \cos^2 A;$$

$$\therefore\ \sin A + \cos A = \pm \sqrt{1 + \sin 2A}.$$

Also, $$1 - \sin 2A = \sin^2 A - 2 \sin A \cos A + \cos^2 A;$$

$$\therefore\ \sin A - \cos A = \pm \sqrt{1 - \sin^2 A}.$$

Hence $$\sin A = \tfrac{1}{2}\{\pm \sqrt{1 + \sin 2A} \pm \sqrt{1 - \sin 2A}\}.$$

And $$\cos A = \tfrac{1}{2}\{\pm \sqrt{1 + \sin 2A} \mp \sqrt{1 - \sin 2A}\}.$$

26. Expressions for the Cosine of twice an Angle in terms of the Cosine and the Sine of the Angle.

$$\cos(A + B) = \cos A \,.\, \cos B - \sin A \,.\, \sin D.$$

Let $B = A$,

then $$\cos(A + A), \text{ or } \cos 2A = \cos^2 A - \sin^2 A.$$

But $$1 = \cos^2 A + \sin^2 A;$$

$$\therefore\ 1 + \cos 2A = 2\cos^2 A, \text{ and, } 1 - \cos 2A = 2 \sin^2 A.$$

27. Formulæ connecting the Sine and Cosine of an Angle with the Sine and Cosine of its half.—If in the formulæ of the last two Articles we write $\frac{A}{2}$ for A, we get

$$\sin A = 2 \sin \frac{A}{2} \,.\, \cos \frac{A}{2},$$

$$1 + \cos A = 2 \cos^2 \frac{A}{2},$$

$$1 - \cos A = 2 \sin^2 \frac{A}{2}.$$

28. Formulæ for the Tangent of half an Angle.

$$\frac{1-\cos A}{\sin A}=\frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\tan\frac{A}{2}.$$

Also,

$$\sqrt{\frac{1-\cos A}{1+\cos A}}=\sqrt{\frac{2\sin^2\frac{A}{2}}{2\cos^2\frac{A}{2}}}=\sqrt{\tan^2\frac{A}{2}}=\tan\frac{A}{2}.$$

29. Tangent of 15°.—In formula $\tan\frac{A}{2}=\frac{1-\cos A}{\sin A}$ substitute 30° for A;

then $$\tan 15°=\frac{1-\cos 30}{\sin 30}=\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=2-\sqrt{3}.$$

30. Tangent of 22½°.—In formula $\tan\frac{A}{2}=\frac{1-\cos A}{\sin A}$ substitute 45° for A;

then $$\tan 22\tfrac{1}{2}°=\frac{1-\cos 45°}{\sin 45°}=\frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}=\sqrt{2}-1.$$

31. Expressions for the Tangent of the Sum and Tangent of the Difference of two Angles in terms of the Tangents of the Angles themselves.

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\,.\,\sin B}{\cos A\,.\,\cos B-\sin A\,.\,\sin B}.$$

Dividing each term of the numerator and denominator by $\cos A \cdot \cos B$,

$$\tan (A + B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A \cdot \sin B}{\cos A \cdot \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}. \qquad (\alpha)$$

Similarly it may be shown that

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}. \qquad (\beta)$$

32. **Formula for the Tangent of twice an Angle in terms of the Tangent of the Angle itself.**

In formula (α, Art. 31), let $B = A$.

Then $$\tan (A + A) \text{ or } \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

33. **Formulæ for the Sum and Difference of the Sines**—By first adding, and then subtracting, the first two of the fundamental formulæ, we get

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cdot \cos B, \qquad (\gamma)$$

$$\sin (A + B) - \sin (A - B) = 2 \cos A \cdot \sin B. \qquad (\delta)$$

Let $$A + B = A',$$

and $$A - B = B';$$

therefore $$2A = A' + B' : \quad \therefore A = \frac{A' + B'}{2},$$

$$2B = A' - B'; \quad \therefore B = \frac{A' - B'}{2}.$$

Substituting these values of $A+B$, $A-B$, A, B, in equations (γ) and (δ), we get

$$\sin A' + \sin B' = 2 \sin \frac{A'+B'}{2} \,.\, \cos \frac{A'-B'}{2},$$

and

$$\sin A' - \sin B' = 2 \cos \frac{A'+B'}{2} \,.\, \sin \frac{A'-B'}{2}.$$

Hence

The sum of the sines of any two angles is equal to twicc the sine of half their sum multiplied by the cosine of half their difference.

And

The difference of the sines of any two angles is equal to twice the cosine of half their sum multiplied by the sine of half their difference.

34. **Formulæ for the Sum and Difference of the Cosines.**—By adding and subtracting the third and fourth of the fundamental formulæ, we get

$$\cos (A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B,$$

$$\cos (A-B) = \cos A \,.\, \cos \mathrm{B} + \sin A \,.\, \sin B.$$

Adding,

$$\cos (A+B) + \cos (A-B) = 2 \cos A \,.\, \cos B.$$

Subtracting,

$$\cos (A+B) - \cos (A-B) = -2 \sin A \,.\, \sin B.$$

Substituting, as in last Article, we get

$$\cos A' + \cos B' = 2 \cos \frac{A'+B'}{2} \,.\, \cos \frac{A'-B'}{2},$$

and

$$\cos A' - \cos B' = -2 \sin \frac{A'+B'}{2} \,.\, \sin \frac{A'-B'}{2}.$$

Hence

The sum of the cosines of any two angles is equal to twice the cosine of half their sum multiplied by the cosine of half their difference.

And

The difference of the cosines of any two angles is equal to **Minus** *twice the sine of half their sum multiplied by the sine of half their difference.*

35. Ratio of the Sum of the Sines of two Angles to the Difference of the Sines.

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \cdot \sin \frac{A-B}{2}} \qquad \text{(Art. 33)}$$

$$= \tan \frac{A+B}{2} \cdot \cot \frac{A-B}{2}$$

$$= \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}},$$

or $\sin A + \sin B : \sin A - \sin B = \tan \frac{A+B}{2} : \tan \frac{A-B}{2}$.

36. Formula for the Sine of three times an Angle.

$$\sin 3A = \sin (A + 2A)$$

$$= \sin A \cdot \cos 2A + \cos A \cdot \sin 2A$$

$$= \sin A (\cos^2 A - \sin^2 A) + \cos A \cdot 2 \sin A \cos A$$

(Arts. 25 and 26)

$$= 3 \sin A \cos^2 A - \sin^3 A$$

$$= 3 \sin A (1 - \sin^2 A) - \sin^3 A \qquad \text{(Art. 15)}$$

$$= 3 \sin A - 4 \sin^3 A.$$

37. Formula for the Cosine of three times an Angle.

$$\begin{aligned}\cos 3A &= \cos (A + 2A)\\ &= \cos A \,.\, \cos 2A - \sin A \,.\, \sin 2A\\ &= \cos A\,(\cos^2 A - \sin^2 A) - \sin A \,.\, 2 \sin A \cos A \qquad \text{(Arts. 25 and 26)}\\ &= \cos^3 A - 3 \cos A \,.\, \sin^2 A\\ &= \cos^3 A - 3 \cos A\,(1 - \cos^2 A) \qquad \text{(Art. 15)}\\ &= 4 \cos^3 A - 3 \cos A.\end{aligned}$$

38. Trigonometrical Functions of 18°.

If $A = 18°$, then $3A + 2A = 90°$;

$$\therefore \ \sin 2A = \cos 3A; \qquad \text{(Art. 14)}$$

$$\begin{aligned}\therefore \ 2 \sin A \,.\, \cos A &= 4 \cos^3 A - 3 \cos A,\\ 2 \sin A &= 4 \cos^2 A - 3,\\ 2 \sin A &= 4\,(1 - \sin^2 A) - 3,\\ 4 \sin^2 A + 2 \sin A - 1 &= 0.\end{aligned}$$

Solving which quadratic equation, we get

$$\sin A = \frac{\pm\sqrt{5} - 1}{4}.$$

Since the sin of 18° is positive, we must take the upper sign;

$$\therefore \ \sin 18° = \frac{\sqrt{5} - 1}{4}.$$

$$\cos 18° = \sqrt{1 - \sin^2 18°} = \sqrt{1 - \frac{6 - 2\sqrt{5}}{16}} = \frac{\sqrt{10 + 2\sqrt{5}}}{4},$$

$$\tan 18° = \frac{\sin 18°}{\cos 18°} = \frac{\sqrt{5} - 1}{\sqrt{10 + 2\sqrt{5}}}$$

39. **Trigonometrical Functions of 36°.**

$$\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{1+\sqrt{5}}{4},$$

$$\sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \frac{\sqrt{10-2\sqrt{5}}}{4},$$

$$\tan 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}.$$

40. **Formula for the Tangent of three times an Angle.**

$$\tan 3A = \tan(A + 2A)$$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \cdot \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \cdot \dfrac{2\tan A}{1-\tan^2 A}}. \qquad \text{(Art. 32)}$$

Multiply both numerator and denominator by $1 - \tan^2 A$.

Then $$\tan 3A = \frac{\tan A - \tan^3 A + 2\tan A}{1 - \tan^2 A - 2\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Exercises.

1. Find expressions for the sine, cosine, &c., of 9° (see Article 25).
2. From the consideration that 75° = 45° + 30°, calculate the trigonometrical ratios of 75°.

CHAPTER IV.

41. **Solution of Right-angled Triangles.**—In every triangle there are six elements, viz., the three sides and the three angles, of which any three *independent* quantities being given, the three remaining may be found.

The three angles are not independent; because their sum is always = 180°.

We shall begin with right-angled triangles, denoting in all cases the three angles by A, B, C, of which C is the right angle, and the opposite sides by a, b, c.

Then

$$\sin A = \frac{a}{c} = \cos B; \quad \therefore \; a = c \sin A, \text{ or } c \cos B,$$

$$\tan A = \frac{a}{b} = \cot B; \quad \therefore \; a = b \tan A, \text{ or } b \cot B,$$

$$\sec A = \frac{c}{b} = \operatorname{cosec} B; \; \therefore \; c = b \sec A, \text{ or } b \operatorname{cosec} B.$$

These relations comprise every *trigonometrical* property of the right-angled triangle, and may be stated as follows:—

RULE 1.—*In any right-angled triangle either side is equal to the sine of the opposite angle, or the cosine of the adjacent angle multiplied by the hypotenuse.*

RULE 2.—*In any right-angled triangle either side is equal to the tangent of the opposite, or the cotangent of the adjacent, angle multiplied by the other side.*

RULE 3.—*In any right-angled triangle the hypotenuse is equal to either side multiplied by the secant of the angle adjacent to, or the cosecant of the angle opposite to, that side.*

The student will observe that there is also the *geometrical* relation $c^2 = a^2 + b^2$.

Examples.

1. Given $c = 480$, $A = 53°\,8'$, to find a, b, and B.

Then $a = c \sin A$; $\therefore \log a = \log c + \log \sin A - 10$,

$$\begin{aligned}\log a &= \log 480 + \log \sin 53°\,8' - 10\\ &= 2{\cdot}68124 + 9{\cdot}90311 - 10\\ &= 2{\cdot}58435;\\ \therefore\ a &= 384.\end{aligned}$$

Also $b = c\,.\cos A$; $\therefore \log b = \log c + \log \cos A - 10$

$$\begin{aligned}&= \log 480 + \log \cos 53°\,8' - 10\\ &= 2{\cdot}68124 + 9{\cdot}77812 - 10\\ &= 2{\cdot}45936;\\ \therefore\ b &= 288.\end{aligned}$$

$$\angle B = 90° - A = 90 - 53°\,8' = 36°\,52'.$$

2. Given $c = 98$, $a = 82{\cdot}01$, find b, A, B.

$\sin A = \frac{a}{c}$; $\therefore \log \sin A = 10 + \log a - \log c$

$$\begin{aligned}&= 10 + \log 82{\cdot}01 - \log 98\\ &= 10 + 1{\cdot}91387 - 1{\cdot}99123;\\ \therefore\ A &= 56°\,48',\\ B &= 90 - A = 90° - 56°48' = 33°\,12'.\end{aligned}$$

Also $b = c\,.\cos A$; $\therefore \log b = \log c + \log \cos A - 10$

$$\begin{aligned}&= \log 98 + \log \cos 56°\,48' - 10\\ &= 1{\cdot}99123 + 9{\cdot}73843 - 10\\ &= 1{\cdot}72966;\\ \therefore\ b &= 53{\cdot}66.\end{aligned}$$

CHAPTER V.

42. **Expression for the Sine of the Angle in any Segment of a Circle.**—*In any circle, the chord divided by the diameter is equal to the sine of the angle subtended by the chord at any point on the circumference.*

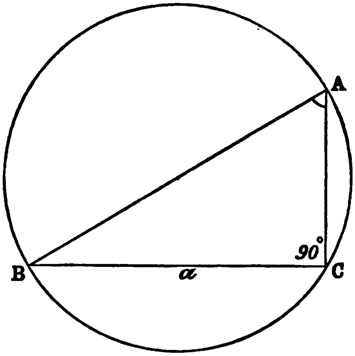

Fig. 9.

Let BC be any chord of a circle.

Through B draw the diameter, meeting the circle in A: join AC; then $\angle BCA = 90°$; (Euc. 31, III.)

$$\therefore \frac{BC}{BA} = \text{sin of } \angle CAB.$$

Now the angle subtended by the chord BC at the point A is either equal to the angle subtended by BC at any other point on the circumference (Euc. 21, III.), or its supplement (Euc. 22, III.); therefore the sine of the angle subtended by BC at any point on the circumference is equal the sine of the angle CAB, equal the chord BC divided by the diameter of the circle.

Cor.—If a equal the length of the chord, and D equal the diameter of the circle, then

$$\frac{a}{D} = \sin A; \quad \therefore \quad D = \frac{a}{\sin A}.$$

43. *The sides of any triangle are proportional to the sines of the opposite angles* (called "Rule of Sines").

Let a, b, c be the three sides of a triangle, A, B, C, the opposite angles, and D the diameter of the circle circumscribing the triangle.

Then $$\frac{a}{\sin A} = D, \ \frac{b}{\sin B} = D, \text{ and } \frac{c}{\sin C} = D;$$

(Art. 42, *Cor.*)

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

or $$a : b : c = \sin A : \sin B : \sin C.$$

Otherwise thus:

If p equal the perpendicular let fall from the angle C on the opposite side, then

$$a \sin B = p = b \sin A; \qquad \text{(Art. 41)}$$

$$\therefore \quad a : b = \sin A : \sin B.$$

44. *The sum of two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference:*

$$a : b = \sin A : \sin B. \qquad \text{(Art. 43)}$$

Composition and division:

$$a+b:a-b=\sin A+\sin B:\sin A-\sin B$$

$$=\tan\frac{A+B}{2}:\tan\frac{A-B}{2}. \qquad \text{(Art. 35)}$$

45. *The sum of two sides of a triangle is to the base, or third side, as the cosine of half the difference of the base angles is to the cosine of half their sum.*

$$a:b:c=\sin A:\sin B:\sin C;$$

$$\therefore\ a+b:c=\sin A+\sin B:\sin C$$

$$=\sin A+\sin B:\sin(A+B) \qquad \text{(Art. 22)}$$

$$=2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}:2\sin\frac{A+B}{2}\cos\frac{A+B}{2}$$

$$=\cos\frac{A-B}{2}:\cos\frac{A+B}{2}.$$

46. *The difference of two sides of a triangle is to the base, or third side, as the sine of half the difference of the base angles is to the sine of half their sum.*

$$\frac{a-b}{c}=\frac{\sin A-\sin B}{\sin C}=\frac{\sin A-\sin B}{\sin(A+B)}$$

$$=\frac{2\cos\frac{A+B}{2}.\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A+B}{2}}$$

$$=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}.$$

47. To express the Cosines of the Angle of a Triangle in terms of the three Sides.

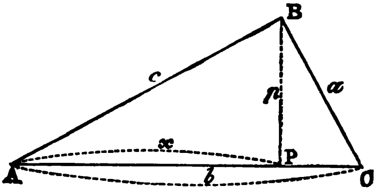

Fig. 10.

From B draw $BP \perp AC$. Let $AP = x$ and $BP = p$.

Then $$a^2 = b^2 + c^2 - 2b \,.\, x. \qquad \text{(Euc. 13, II.)}$$

But $$x = c \cos A; \qquad \text{(Art. 41)}$$

$$\therefore \quad a^2 = b^2 + c^2 - 2b\,.\,c\,.\cos A.$$

$$2b\,.\,c\,.\cos A = b^2 + c^2 - a^2;$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Similarly,
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

48. Area of a Triangle in terms of two Sides and the included Angle.

$$b\,.\,p = 2 \text{ area of } \triangle.$$

But $$p = c\,.\sin A; \qquad \text{(Art. 41)}$$

$$\therefore \quad b.c \sin A = 2 \text{ area of } \triangle;$$

$$\therefore \quad \tfrac{1}{2}\, b.c.\sin A = \text{area of } \triangle.$$

Similarly, $$\text{area} = \tfrac{1}{2}.a.c.\sin B = \tfrac{1}{2}.a.b.\sin C.$$

49. Cosine of Half an Angle of a Triangle in terms of the Three Sides.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

$$\therefore \quad 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc},$$

$$2\cos^2 \frac{A}{2} = \frac{2bc + b^2 + c^2 - a^2}{2bc},$$

$$= \frac{(b + c)^2 - a^2}{2bc},$$

$$= \frac{(b + c + a)(b + c - a)}{2bc}.$$

Let s = the semi-perimeter of the triangle or half the sum of the three sides,

then $a + b + c = 2s,$

and $a + b - c = a + b + c - 2c = 2(s - c).$

Similarly $a - b + c = 2(s - b),$

and $b + c - a = 2(s - a);$

$$\therefore \quad 2\cos^2 \frac{A}{2} = \frac{2s \cdot 2(s - a)}{2bc},$$

and $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s - a)}{b \cdot c}}.$

50. Sine of Half an Angle of a Triangle in terms of the Three Sides.

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc},$$

$$2\sin^2\frac{A}{2}=\frac{2bc-(b^2+c^2-a^2)}{2bc},$$

$$=\frac{a^2-(b^2+c^2-2bc)}{2bc},$$

$$=\frac{a^2-(b-c)^2}{2bc},$$

$$=\frac{(a+b-c)\,(a-b+c)}{2bc}.$$

Substituting as above; then

$$2\sin^2\frac{A}{2}=\frac{2(s-c)\;2(s-b)}{2bc};$$

$$\therefore\quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

51. Tangent of Half an Angle of a Triangle in terms of the Three Sides.

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

52. Sine of an Angle of a Triangle in terms of the Three Sides.

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}=2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}};$$

$$\therefore\quad \sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

53. Area of a Triangle in terms of the three Sides.

$$\text{area of } \triangle = \tfrac{1}{2}bc\sin A, \qquad \text{(Art. 48)}$$

$$= \tfrac{1}{2}.b.c.\frac{2}{bc}\sqrt{s.(s-a)(s-b)(s-c)} \qquad \text{(Art. 52)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

54. Diameter of the Circle circumscribing a Triangle in terms of the Sides.

$$D = \frac{a}{\sin A}, \qquad \text{(Art. 42)}$$

$$\text{area of } \triangle = \frac{bc\sin A}{2}; \qquad \text{(Art. 48)}$$

$$\therefore \quad D \times \text{area of } \triangle = \frac{abc}{2};$$

$$\therefore \quad D = \frac{abc}{2\,\text{area of } \triangle}.$$

55. Radii of Inscribed and Escribed Circles.

Let r = radius of inscribed circle :

then

$$ra = 2\,\text{area of } \triangle\ COB,$$
$$rb = 2\,\text{area of } \triangle\ COA,$$
$$rc = 2\,\text{area of } \triangle\ AOB;$$
$$\therefore \quad r(a+b+c) = 2\,\text{area of } \triangle\ ABC;$$
$$\therefore \quad r = \frac{2\,\text{area of } \triangle\ ABC}{2s} = \frac{\text{area of } \triangle}{s}.$$

Let r' = radius of an escribed circle :

then

$$r'b = 2\,\text{area of } \triangle\ CO'A,$$
$$r'c = 2\,\text{area of } \triangle\ BO'A,$$
$$r'a = 2\,\text{area of } \triangle\ CO'B;$$

$$\therefore\ r'(b+c-a) = 2 \text{ area of trap } CO'BA - 2 \text{ area of } \triangle\ CO'B$$
$$= 2 \text{ area of } \triangle\ ABC;$$

$$\therefore\qquad r' = \frac{2 \text{ area of } \triangle}{2(s-a)} = \frac{\text{area}}{s-a}.$$

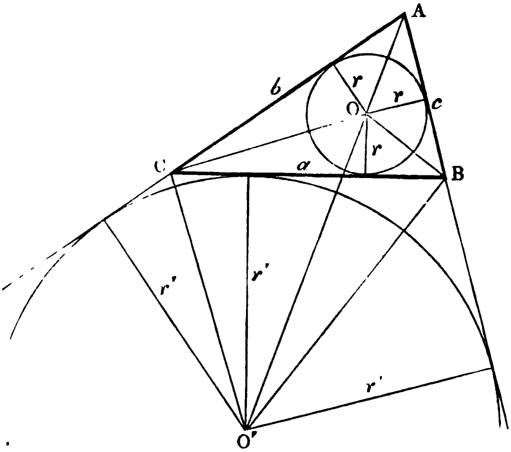

Fig. 11.

Cor. 1.—If r'' and r''' be the radii of the other escribed circles, similarly

$$r'' = \frac{\text{area}}{s-b}, \quad \text{and } r''' = \frac{\text{area}}{s-c}.$$

Cor. 2.—
$$\frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''} = \frac{s-a}{\text{area}} + \frac{s-b}{\text{area}} + \frac{s-c}{\text{area}}$$
$$= \frac{3s-(a+b+c)}{\text{area}}$$
$$= \frac{3s-2s}{\text{area}}$$
$$= \frac{s}{\text{area}} = \frac{1}{r}.$$

Cor. 3.— $$r\,.\,r'\,.\,r''\,.\,r''' = \frac{\text{area}}{s}\,.\,\frac{\text{area}}{s-a}\,.\,\frac{\text{area}}{s-b}\,.\,\frac{\text{area}}{s-c}$$

$$= \frac{(\text{area})^4}{s\,(s-a)\,(s-b)\,(s-c)}$$

$$= \frac{\text{area}^4}{\text{area}^2} = \text{area}^2;$$

$$\therefore\quad \text{area} = \sqrt{r\,.\,r'\,.\,r''\,.\,r'''}.$$

56. **To find the Area of a Quadrilateral inscribed in a Circle.**

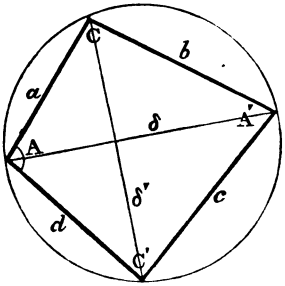

Fig. 12.

$$a^2 + b^2 - 2\,ab\cos C = \delta^2 = c^2 + d^2 - 2\,cd\cos C'. \qquad \text{(Art. 47)}$$

Since $C + C' = 180°$, $\cos C' = -\cos C$;

therefore $$\cos C = \frac{a^2 + b^2 - c^2 - d^2}{2\,(ab + cd)};$$

therefore

$$\sin^2 C = 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{4\,(ab + cd)^2} = \frac{4\,(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2}{4\,(ab + cd)^2}$$

$$= \frac{\{2\,(ab + cd) + c^2 + d^2 - a^2 - b^2\}\,.\,\{2\,(ab + cd) - c^2 - d^2 + a^2 + b^2\}}{4\,(ab + cd)^2}$$

$$= \frac{\{(c+d)^2-(a-b)^2\}\{(a+b)^2-(c-d)^2\}}{4(ab+cd)^2}$$

$$= \frac{(c+d+a-b)(c+d-a+b)(a+b+c-d)(a+b-c+d)}{4(ab+cd)^2}.$$

Let $a+b+c+d = 2s;$

then $a+b+c-d = 2(s-d)$, &c.;

$$\therefore \quad \sin C = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}.$$

Now, area of quadrilateral $= \dfrac{ab\sin C}{2} + \dfrac{cd\sin C'}{2}$

(Art. 48)

$$= (ab+cd)\frac{\sin C}{2}$$

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

57. **To find the Diameter of a Circle in terms of the Sides of an Inscribed Quadrilateral.**—Let D denote the diameter of a circle; then

$$\sin C = \frac{\delta}{D} = \sin C', \quad \text{and } \sin A = \frac{\delta'}{D} = \sin A'; \quad \text{(Art.42)}$$

$$\therefore \quad (ab+cd)\frac{\delta}{D} = 2\,\text{area of quadrilateral} = (bc+ad)\frac{\delta'}{D}.$$

Also

$$ac+bd = \delta\,.\,\delta'; \qquad \text{(Ptolemy's Theorem.)}$$

$$\therefore \quad ac+bd = \frac{2D}{ab+cd} \times \frac{2D}{bc+ad} \times (\text{area of quadrilateral})^2;$$

$$\therefore \quad \frac{(ac+bd)(ab+cd)(bc+ad)}{4\,(\text{area of quadrilateral})^2} = D^2;$$

$$\therefore \quad D = \tfrac{1}{2}\sqrt{\frac{(ac+bd)(ab+cd)(bc+ad)}{(s-a)\,.\,(s-b)\,.\,(s-c)\,.\,(s-d)}}.$$

Cases of Oblique-angled Triangles.

58. **First Case.**—Given two angles (A, B) and the side (a) opposite one of them:

$$C = 180° - (A + B),$$

$$\sin A : \sin B = a : b; \qquad \therefore\ b = \frac{a \sin B}{\sin A}$$

$$\sin A : \sin C, \text{ or } \sin (A + B) = a : c; \quad \therefore\ c = \frac{a \sin (A + B)}{\sin A}.$$

Second Case.—Give two angles (A, B) and the adjacent side (c):

$$C = 180 - (A + B),$$

$$\sin C : \sin A = c : a; \quad \therefore\ a = \frac{c \sin A}{\sin C},$$

$$\sin C : \sin B = c : b; \quad \therefore\ b = \frac{c \sin B}{\sin C}.$$

Third Case.—Given two sides (a, b) and the angle (A) opposite one of them:

$a : b = \sin A : \sin B$, from which we determine $\sin B$.

If $\frac{b}{a} . \sin A$ is < 1, two different angles may be found, the one $< 90°$ and the other $> 90°$ and $< 180°$, which have $\frac{b}{a} . \sin A$ for their sine.

If a be $> b$, then angle A must be $> B$, and therefore angle B *must* be the less of the two values found.

If $a < b$, then either of the values may be taken for B. *Hence, when two sides and the angle opposite the less are given, there can usually be found two triangles from the given parts, and this is called the ambiguous case of plane triangles.* We say *usually*, for if $\frac{b}{a} \sin A = 1$, then $B = 90°$, and only one triangle can be found. Should $\frac{b \sin A}{a}$ be > 1, it is impossible to construct a triangle with the data.

Also $$180 - (A + B) = C.$$

And $$\sin A : \sin C = a : c; \quad \therefore\ c = \frac{a \sin C}{\sin A}.$$

Fourth Case.—Given two sides (a, b) and the included angle:

$$180 - C = A + B$$

$$a + b : a - b = \tan\frac{A+B}{2} : \tan\frac{A-B}{2};$$

$$\therefore\ \tan\frac{A-B}{2} = \frac{a-b}{a+b} \cdot \tan\frac{A+B}{2}.$$

This determines $\frac{A-B}{2}$; but $A + B$ is known; hence A and B can be immediately found.

Also $\sin A : \sin C = a : c$; $\therefore\ c = \frac{a \sin C}{\sin A}$, from which c is found.

c may, however, in practice, be more conveniently determined by formulæ of Articles 45 and 46.

$$c = \frac{(a+b) \cdot \cos\frac{A+B}{2}}{\cos\frac{A-B}{2}}, \quad \text{or} \quad c = \frac{(a-b)\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}}.$$

The third side might be found directly by the formula $a^2 = b^2 + c^2 - 2\,bc\cos A$; the objection to the use of which formula is, that it is not adapted to logarithmic computation. It may, however, be adapted to logarithms, *see* p. 55.

Fifth Case.—Given the three sides (a, b, c). The angles may be calculated by any of the following formulæ—

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc};$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}};$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}};$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

$$\sin A = \frac{2}{bc}\sqrt{s\,.\,(s-a)(s-b)(s-c)}.$$

The formula for the tan $\frac{A}{2}$, which may be shown to be

$$= \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

is, however, the most convenient in practice.

Exercises.

1. Show from geometrical considerations that there is an ambiguity when two sides of a triangle and the angle opposite the less are given.

2. Show that the circles circumscribing the two triangles are equal.

3. If $\angle A = 27^\circ\, 59'$, $\angle B = 44^\circ\, 12'$, and $b = 208$; find a, c, and C.
Ans. $a = 140$, $c = 284{\cdot}02$; $C = 107^\circ\, 49'$.

4. If $a = 145{\cdot}3$ $b = 178{\cdot}3$ and $A = 41^\circ\, 10'$; find the other parts.
Ans. $B = 53^\circ\, 52'\, 36''$, or $126^\circ\, 7'\, 24''$;
$C = 84^\circ\, 57'\, 24''$, or $12^\circ\, 42'\, 36''$;
$c = 219{\cdot}882$, or $48{\cdot}5656$.

5. Given $a = 190{\cdot}24$, $b = 324{\cdot}68$, $c = 196$. Calculate the angles.
Ans. $A = 32^\circ\, 15'$, $B = 114^\circ\, 24'$, $C = 33^\circ\, 21'$.

6. Given $a = 224$, $b = 240$, $c = 225{\cdot}3$. Calculate the angles.
Ans. $A = 57^\circ\, 27'$, $B = 64^\circ\, 34'$, $C = 57^\circ\, 59'$.

7. Find the angles of the triangle whose sides are 5, 6, 7.
Ans. $44^\circ\, 24'\, 53''$, $57^\circ\, 7'\, 18''$, $78^\circ\, 27'\, 49''$.

8. If in a triangle $A = 26^\circ\, 38'\, 50''$, $B = 34^\circ\, 32'\, 13''$, $c = 170$; find a and b.
Ans. 87·02, 110.

9. If in a triangle $a = 6$, $b = 7$, $C = 75^\circ\, 31'\, 22''$; find c, A, B.
Ans. 8, $46^\circ\, 34'\, 3''$, $57^\circ\, 54'\, 35''$.

10. Show that $c = a \cos B + b \cos A$.

11. If r = the radius of a circle about which any polygon is circumscribed, show that area of polygon $= r \times$ semi-perimeter of the polygon.

12. Hence show that area of a circle $= \pi r^2$.

13. Given $a = 1052{\cdot}627$, $b = 1092{\cdot}982$, $B = 63^\circ\, 42'\, 25''$; find the other parts.
Ans. $c = 1017{\cdot}63$ $A = 56^\circ\, 35'\, 18''$;
$C = 59^\circ\, 42'\, 17''$.

14. If θ be the circular measure of the angle of a sector of a circle, show that area of sector $= \dfrac{r^2\theta}{2}$.

15. Hence show that the area of a segment of a circle $= \dfrac{r^2}{2}(\theta - \sin\theta)$.

CHAPTER VI.

59. **Definition of Logarithm.**—The logarithm of a number is the exponent of the power to which a given base must be raised so as to be equal to the number.

Thus, if $$a^x = P,$$

x is called the logarithm of P to the base a, and is written

$$x = \log_a P.$$

Since $$a^0 = 1; \quad \therefore\ 0 = \log_a 1.$$

That is, *The logarithm of unity in any system is equal to zero.*

Also $$a^1 = a; \quad \therefore\ 1 = \log_a a;$$

or, *The logarithm of the base is equal to unity.*

$$a^\infty = \infty; \quad \therefore\ \infty = \log \infty.$$

The logarithm of infinity is equal to infinity.

$$\frac{1}{0} = \infty;$$

$$\begin{aligned} \therefore\quad \log 1 - \log 0 &= \log \infty, \\ 0 - \log 0 &= \infty; \\ \therefore\quad \log 0 &= -\infty. \end{aligned}$$

The logarithm of zero is equal to minus infinity.

Let $a^x = P$; then $x = \log_a P$,

and $a^y = Q$, $y = \log_a Q$;

$\therefore \quad a^{x+y} = P \,.\, Q$, and $x + y = \log_a P + \log_a Q$;

$\therefore \quad x + y = \log_a P \,.\, Q$.

Hence $\log_a P + \log_a Q = \log_a P \,.\, Q$.

That is, *The sum of the logarithms of two numbers is equal to the logarithm of their product.*

Again, if $a^x = P$, then $x = \log_a P$,

and $a^y = Q$, $y = \log_a Q$;

$\therefore \quad a^{x-y} = \dfrac{P}{Q}$; and $x - y = \log_a P - \log_a Q$;

$\therefore \quad x - y = \log_a \dfrac{P}{Q}$.

Hence $\log_a P - \log_a Q = \log_a \dfrac{P}{Q}$.

That is, *The difference of the logarithms of two numbers is equal to the logarithm of their quotient.*

If $a^x = P$, then $x = \log_a P$.

$a^{nx} = P^n$, $nx = n \log_a P$;

$\therefore \quad nx = \log_a P^n$;

$\therefore \quad \log_a P^n = n \log_a P$.

That is, *The logarithm of the n^{th} power of a number is equal to n times the logarithm of that number.*

If $a^x = P$, then $x = \log_a P$,

$$a^{\frac{x}{n}} = P^{\frac{1}{n}} = \sqrt[n]{P}, \qquad \frac{x}{n} = \frac{1}{n}\log_a P;$$

$$\therefore\ \frac{x}{n} = \log_a \sqrt[n]{P}.$$

Hence $$\log_a \sqrt[n]{P} = \frac{1}{n}\log_a P.$$

That is, *The logarithm of the n^{th} root of a number is equal to the $\frac{1}{n^{th}}$ part of the logarithm of that number.*

If $b^x = R$, then $x = \log_b R$.

Also take the logarithm of both sides to base a.

Then $$x \log_a b = \log_a R;$$

$$\therefore\ x = \frac{\log_a R}{\log_a b};$$

$$\therefore\ \log_b R = \frac{\log_a R}{\log_a b}.$$

By the aid of this formula, knowing the logarithms of numbers to any base a, their logarithms to any other base b may be found.

The logarithms generally used are those calculated to the base 10, and are called *Common Logarithms.*

The decimal part only of the logarithm of a number, which is called the *mantissa,* is given in the Tables; the integral part, called the *characteristic,* can be determined by inspection.

Since $10^1 = 10$; $\therefore$ $1 = \log 10$.
$10^2 = 100$; $\therefore$ $2 = \log 100$.
$10^3 = 1{,}000$; $\therefore$ $3 = \log 1{,}000$.
$10^4 = 10{,}000$; $\therefore$ $4 = \log 10{,}000$.

Hence the logarithms of all numbers between 10 and 100 lie between 1 and 2; the logs of all numbers between 100 and 1,000 lie between 2 and 3; the logs of all numbers between 1,000 and 10,000 lie between 3 and 4; and, in general, *the characteristic has in it as many units as there are digits in the integral part of the number minus one.*

Tables of natural sines, cosines, &c., of all angles from 0 to 90° have been constructed, but as the sines and cosines of all angles are either unity, or fractions less than unity, the logarithms of these sines and cosines are negative. The same may be said of the tangents of all angles less than 45°. To avoid negative indices, therefore, 10 is added to each logarithmic, sine, cos, tangent, &c., before it is entered in the Tables.

Exercises.

1. Given $\log_{10} 2 = 0{\cdot}3010300$ and $\log_{10} 3 = 0{\cdot}4771213$. Find the logarithms of 5, 6, 9, 125, and ·015.

2. Find the fifth root of ·0625, being given $\log_{10} 2 = 0{\cdot}3010300$ and $\log_{10} 5{\cdot}743491 = 0{\cdot}7591760$.

3. If
$$7^x = \frac{3{\cdot}45^{\frac{2}{3}} \times 86{\cdot}2^{\frac{5}{6}}}{{\cdot}068^{\frac{3}{2}}},$$
write down the logarithmic equation for determining x.

CHAPTER VII.

61. **Heights and Distances.**—*To find the height and distance of an inaccessible object on a horizontal plane.*

Measure a straight line AB on the horizontal plane in a direct line towards the base of the object. At A and B observe the angles of elevation CAD, CBD, of the top.

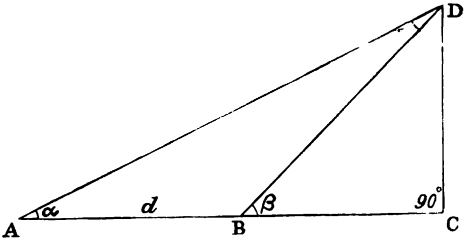

Fig. 13.

Let $AB = d$, $\angle CAD = \alpha$, $\angle CBD = \beta$; then $ADB = \beta - \alpha$.

Then $\sin(\beta - \alpha) : \sin \alpha :: d : BD$; $\therefore\ BD = \dfrac{d \sin \alpha}{\sin(\beta - \alpha)}$.

$$\text{Also}\quad CD = BD \,.\, \sin \beta = \frac{d \sin \alpha \,.\, \sin \beta}{\sin(\beta - \alpha)}.$$

$$\text{And}\quad BC = BD \,.\, \cos \beta = \frac{d \sin \alpha \,.\, \cos \beta}{\sin(\beta - \alpha)}.$$

Example.

Let $d = 75$ feet, $\beta = 51° 30'$, $\alpha = 26° 30'$.

$\log CD = \log 75 + \log \sin 26° 30' + \log \sin 51° 30' - \log \sin 25° - 10,$

$\log BC = \log 75 + \log \sin 26° 30' + \log \cos 51° 30' - \log \sin 25° - 10,$

$$
\begin{array}{lrl}
\log 75 & = & 1{\cdot}87506 \\
\log \sin 26^\circ\,30' & = & 9{\cdot}64953 \\
\log \sin 51^\circ\,30' & = & \underline{9{\cdot}89354} \\
 & & 21{\cdot}41813 \\
10 + \log \sin 25^\circ & = & \underline{19{\cdot}62595} \\
\log CD & = & 1{\cdot}79218 \\
\therefore\quad CD & = & 61{\cdot}97\,;
\end{array}
\qquad
\begin{array}{lrl}
\log 75 & = & 1{\cdot}87506 \\
\log \sin 26^\circ\,30' & = & 9{\cdot}64953 \\
\log \cos 51^\circ\,30' & = & \underline{9{\cdot}79415} \\
 & & 21{\cdot}31874 \\
10 + \log \sin 25 & = & \underline{19{\cdot}62595} \\
\log BC & = & 1{\cdot}69279\,; \\
\therefore\quad BC & = & 49{\cdot}29.
\end{array}
$$

Exercises.

1. If $d = 412{\cdot}78$, $\alpha = 19^\circ\,17'\,20''$, $\beta = 47^\circ\,11'\,12''$, show that $CD = 213{\cdot}7775$, and $BC = 198{\cdot}0526$.

2. Two observers on the same side of a balloon, and in the same vertical plane with it, a mile apart, find its angles of elevation to be respectively $15^\circ\,17'\,18''$ and $62^\circ\,30'\,20''$; find height of balloon.

Ans. Height = 560·89 yds.

3. Prove the formula when the observers are on opposite sides of the balloon.

Ans. $H = \dfrac{d \sin\alpha \,.\, \sin\beta}{\sin(\alpha + \beta)}$.

4. Show that the same formula will give the perpendicular width of a river.

5. Show that breadth of river is also equal to $\dfrac{d}{\cot\alpha + \cot\beta}$.

62. **To determine the Height and Distance of an Object situated on an Eminence, and seen from a Horizontal Plane.**

Let *DE* be the object, *AB* a given line in the horizontal plane, measured in a direct line towards the object.

At A observe the angle of elevation of D, and at B observe the angles of elevation of D and E.

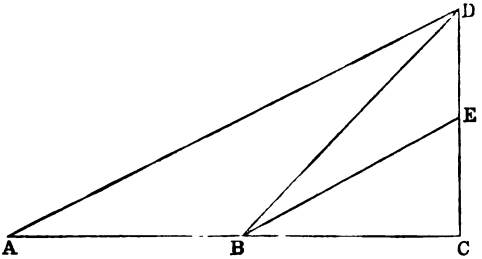

Fig. 14.

Let $AB = d$, $\angle DAB = \alpha$, $DBC = \beta$, $EBC = \gamma$.

Then, as in last Problem, $CD = \dfrac{d \sin\alpha \,.\, \sin\beta}{\sin(\beta - \alpha)}$,

$$BC = \frac{d \sin\alpha \cos\beta}{\sin(\beta - \alpha)}.$$

And therefore $CE = BC \,.\, \tan\gamma = \dfrac{d \sin\alpha \cos\beta \,.\, \tan\gamma}{\sin(\beta - \alpha)}$.

Hence we find DE.

Example.

If $AB = 134$, $\alpha = 44°$, $\beta = 67°50'$, $\gamma = 51°$,

log 134 =	2·12710	log 134 =	2·12710
log sin 44° =	9·84177	log sin 44° =	9·84177
log sin 67° 50′ =	9·96665	log cos 67° 50′ =	9·57669
	21·93552		21·54556
10 + log sin 23° 50′ =	19·60646	10 + log sin 23° 50′ =	19·60646
log CD =	2·32906	log BC =	1·93910
CD =	213·33	BC =	86·917

$$\log CE = \log BC + \log \tan \gamma - 10$$

$$\begin{aligned} \log BC &= 1{\cdot}93910 \\ \log \tan 51^\circ &= 10{\cdot}09163 \\ \hline \log CE &= 12{\cdot}03073 - 10; \\ \therefore\ CE &= 107{\cdot}33. \end{aligned}$$

Hence $DE = DC - CE = 213{\cdot}33 - 107{\cdot}33 = 106.$

63. To find the Distance between Two Inaccessible Objects (A, B) standing on a Horizontal Plane.

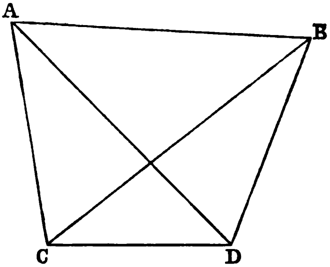

Fig. 15.

Measure a bass line CD; at C observe the angles ACD and BCD, and at D observe the angles ADC and BDC. Let

$CD = d, \angle ACD = \alpha, \angle BCD = \alpha', \angle ADC = \beta$, and $\angle BDC = \beta'$.

In

$\triangle$ CAD $\sin(\alpha + \beta) : \sin\beta = d : AC$; $\therefore AC = \dfrac{d \sin\beta}{\sin(\alpha + \beta)}$.

In

$\triangle$ CBD $\sin(\alpha' + \beta') : \sin\beta' = d : CB$; $\therefore CB = \dfrac{d \sin\beta'}{\sin(\alpha' + \beta')}$.

Now in $\triangle$ ACB we know two sides, and the included $\angle = \alpha - \alpha'$. Hence we can calculate AB by method of Article 58, Fourth Case.

Example.

Let $d = 300$ yards, $\alpha = 95^\circ\ 20'$, $\alpha' = 58^\circ\ 20'$, $\beta = 53^\circ\ 30'$, $\beta' = 98^\circ . 45'$. Find AB.

$$\begin{aligned}
\log 300 &= 2{\cdot}47712 \\
\log \sin 53^\circ\ 30' &= 9{\cdot}90518 \\
\hline
&\ 12{\cdot}38230 \\
\log \sin 31^\circ\ 10' &= 9{\cdot}71393 \\
\therefore \log AC &= 2{\cdot}66837 \\
\therefore AC &= 465{\cdot}97;
\end{aligned}$$

$$\frac{1227{\cdot}43}{295{\cdot}49} = \frac{\tan\ 71^\circ\ 30'}{\tan \dfrac{A-B}{2}}$$

$$\begin{aligned}
\log \tan 71^\circ\ 30' &= 10{\cdot}47548 \\
\log 295{\cdot}49 &= 2{\cdot}47055 \\
\hline
&\ 12{\cdot}94603 \\
\log 1227{\cdot}43 &= 3{\cdot}08893 \\
\hline
\therefore \log \tan\left(\frac{A-B}{2}\right) &= 9{\cdot}85713; \\
\therefore \frac{A-B}{2} &= 35^\circ\ 45';
\end{aligned}$$

$$\begin{aligned}
\log 300 &= 2{\cdot}47712 \\
\log \sin 98^\circ\ 45' &= 9{\cdot}99492 \\
\hline
&\ 12{\cdot}47204 \\
\log \sin 22{\cdot}45 &= 9{\cdot}59039; \\
\log CB &= 2{\cdot}88165; \\
\therefore CB &= 761{\cdot}46.
\end{aligned}$$

$$\frac{\sin 35^\circ\ 44'}{\sin 71^\circ\ 30'} = \frac{295{\cdot}49}{c}.$$

$$\begin{aligned}
\log 295{\cdot}49 &= 2{\cdot}47055 \\
\log \sin 71^\circ\ 30' &= 9{\cdot}97696 \\
\hline
&\ 12{\cdot}44751 \\
\log \sin 35^\circ\ 44' &= 9{\cdot}76642 \\
\hline
\therefore \log c &= 2{\cdot}68109; \\
\therefore c = AB &= 479{\cdot}79.
\end{aligned}$$

64. To find the height of an Object above a Horizontal Plane from Observations made at three Stations in a straight line not in the direction of the Object.

Let H be the object; ABC the three stations at which the angles of elevation are observed.

Denote the angles HAO, HBO, HCO, by α, β, γ, and the distances BC, CA, AB, by a, b, c, and the required height HO by h; then

$$AO = h \cot \alpha; \quad BO = h \cot \beta; \quad CO = h \cot \gamma; \qquad (41)$$

$$\therefore \quad \frac{h^2 \cot^2 \gamma + b^2 - h^2 \cot^2 \alpha}{2bh \cot \gamma} = \cos \angle ACO = - \angle BCO$$

$$= - \frac{h^2 \cot^2 \gamma + a^2 - h^2 \cot^2 \beta}{2ah \cot \gamma};$$

$$\therefore \; ah^2 \cot^2 \gamma + ab^2 - ah^2 \cot^2 \alpha = - bh^2 \cot^2 \gamma - ba^2 + bh^2 \cot^2 \beta,$$

$$h^2 \left(a \cot^2 \alpha + b \cot^2 \beta - (a + b) \cot^2 \gamma\right) = ab\,(a + b);$$

$$\therefore \; h^2 = \frac{abc}{a \cot^2 \alpha + b \cot^2 \beta - c \cot^2 \gamma}.$$

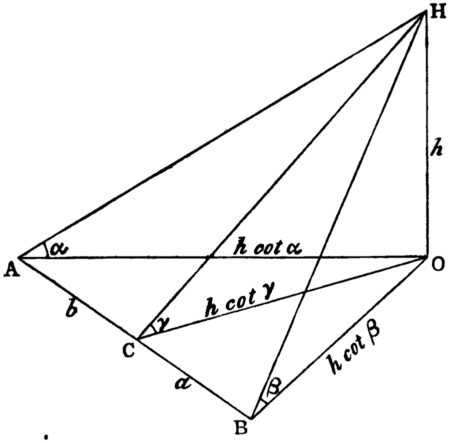

Fig. 16.

CHAPTER VIII.

65. Reduction of Formulæ.

1°. $$\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \frac{2 \sin 2A \cos A}{2 \cos 2A \, . \cos A} \quad \text{(Arts. 33, 34)}$$

$$= \tan 2A.$$

2°.

$$\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \frac{\sin 3A + (\sin A + \sin 5A)}{\cos 3A + (\cos A + \cos 5A)}$$

$$= \frac{\sin 3A + 2 \sin 3A \cos 2A}{\cos 3A + 2 \cos 3A \, . \cos 2A}.$$

(Arts. 33, 34)

$$= \frac{\sin 3A \; (1 + 2 \cos 2A)}{\cos 3A \; (1 + 2 \cos 2A)}$$

$$= \tan 3 \text{A}.$$

3°. $\sin^2 A - \sin^2 B = (\sin A + \sin B)(\sin A - \sin B)$

$$= 2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2} \cdot 2 \cos \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$$

$$= 2 \sin \frac{A+B}{2} \cdot \cos \frac{A+B}{2} \cdot 2 \sin \frac{A-B}{2} \cdot \cos \frac{A-B}{2}$$

$$= \sin (A + B) \, . \sin (A - B).$$

4°. $$\cos(A+B)\,.\,\cos(A-B)$$
$$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\,.\,\sin B)$$
$$= \cos^2 A\cos^2 B - \sin^2 A\,.\,\sin^2 B$$
$$= \cos^2 A\,(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$$
$$= \cos^2 A - \sin^2 B.$$

5°. $$\frac{\cot A + \tan A}{\cot A - \tan A} = \frac{\dfrac{\cos A}{\sin A} + \dfrac{\sin A}{\cos A}}{\dfrac{\cos A}{\sin A} - \dfrac{\sin A}{\cos A}}$$
$$= \frac{\cos^2 A + \sin^2 A}{\cos^2 A - \sin^2 A}$$
$$= \frac{1}{\cos 2A} = \sec 2A.$$

6°. If in a triangle $\dfrac{\tan A}{\tan B} = \dfrac{\sin^2 A}{\sin^2 B}$, prove that the triangle is either isosceles or right-angled. Simplifying, we get

$$\tan A\sin^2 B = \tan B\,.\,\sin^2 A,$$
$$\frac{\sin A}{\cos A}\,.\,\sin^2 B = \frac{\sin B}{\cos B}\,.\,\sin^2 A;$$
$$\therefore\ \sin B\cos B = \sin A\cos A,$$
$$\sin 2B = \sin 2A.$$

Hence either $2B = 2A$ or $2B + 2A = 180°$;

$\therefore$ either $A = B$, or $A + B = 90°$.

7°. The following useful relations connect the three angles of a triangle:

$$A + B + C = 180°;$$

$$\therefore \qquad \tan(A+B) = -\tan C,$$

$$\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = -\tan C;$$

$$\therefore \quad \tan A + \tan B = -\tan C + \tan A \cdot \tan B \cdot \tan C;$$

$$\therefore \quad \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C.$$

8°. Also $\cos(A+B) = -\cos C,$

$$\cos A \cos B - \sin A \cdot \sin B = -\cos C,$$

$$\cos A \cos B + \cos C = \sin A \cdot \sin B.$$

Squaring,

$$\begin{aligned} \cos^2 A \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C &= \sin^2 A \cdot \sin^2 B \\ &= (1-\cos^2 A)(1-\cos^2 B) \\ &= 1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B; \end{aligned}$$

$$\therefore \quad \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cdot \cos B \cdot \cos C = 1.$$

9°. Solve the equation $a\cos x + b\sin x = c.$

$$a\cos x = c - b\sin x,$$

$$a^2\cos^2 x = c^2 - 2bc\sin x + b^2\sin^2 x.$$

$$a^2(1-\sin^2 x) = c^2 - 2bc\sin x + b^2\sin^2 x,$$

$$(a^2+b^2)\sin^2 x - 2bc\sin x + c^2 - a^2 = o;$$

$$\therefore \quad \sin x = \frac{2bc \pm \sqrt{4b^2c^2 - 4(a^2+b^2)(c^2-a^2)}}{2(a^2+b^2)}$$

$$= \frac{bc \pm a\sqrt{a^2+b^2-c^2}}{a^2+b^2}.$$

Otherwise thus: $a \cos x + b \sin x = c,$

$$\cos x + \frac{b}{a} \sin x = \frac{c}{a}.$$

Let $\frac{b}{a} = \tan \phi.$

then $\cos x + \tan \phi \, . \sin x = \frac{c}{a};$

$$\therefore \quad \cos x \cos \phi + \sin x \sin \phi = \frac{c}{a} \cos \phi,$$

$$\cos (x - \phi) = \frac{c}{a} \cos \phi.$$

From which we obtain the value of $x - \phi$; but ϕ is known from the equation $\tan \phi = \frac{b}{a}$ and, therefore we know x.

NOTE.—The angle ϕ thus introduced to facilitate calculation is called a *Subsidiary Angle.*

10°. The formula $a^2 = b^2 + c^2 - 2bc \cos A$. Art. 47 can be reduced to a form adapted to logarithmic computation by means of a subsidiary angle; substituting for $\cos A$ its value $1 - 2 \sin^2 \frac{A}{2}$ (Art. 27),
we get

$$a^2 = b^2 + c^2 - 2bc\left(1 - 2 \sin^2 \frac{A}{2}\right)$$

$$= (b - c)^2 + 4\, bc \sin^2 \frac{A}{2}$$

$$= (b - c)^2 \left\{1 + \frac{4\, bc \sin^2 \frac{A}{2}}{(b - c)^2}\right\}.$$

Let $$\frac{4\,bc\sin^2\dfrac{A}{2}}{(b - c^2)} = \tan^2\phi.$$

Then $$a^2 = (b - c)^2\,(1 + \tan^2\phi);$$

$$\therefore \quad a = (b - c)\,.\,\sec\phi;$$

$$\therefore \quad \log a = \log(b - c) + \log\sec\phi - 10.$$

The equation for determining ϕ gives

$$2\log\tan\phi = \log 4 + \log b + \log c + 2\log\sin\frac{A}{2} - 2\log(b - c).$$

Exercise.

If $\tan\dfrac{\beta - \alpha}{2} = \dfrac{3\sin\alpha}{5 - 3\cos\alpha}$, then $\tan\dfrac{\beta}{2} = 4\tan\dfrac{\alpha}{2}$.

Examples for Practice.

1. Prove geometrically $\dfrac{\sin A}{1+\cos A} = \tan\dfrac{A}{2}$.

2. Prove geometrically $\dfrac{1-\cos A}{\sin A} = \tan\dfrac{A}{2}$.

3. Prove geometrically the formulæ in Articles 44, 45, 46.

4. Show that $\sin 30^\circ\, 5' = \cdot 50126$.

5. Show that $\sin(180^\circ + A) = -\sin A$.

6. If $\tan A = a$, and $\tan B = \dfrac{1}{a}$, show that $A + B = 90^\circ$.

7. The earth, whose diameter = 7920 miles, subtends an angle of 114′ at the moon: calculate the distance of the moon from the earth.

8. Write down the secants of the following angles, viz.—

$$30^\circ,\ 45^\circ,\ 60^\circ,\ 90^\circ,\ 120^\circ,\ 135^\circ,\ 150^\circ,\ 180^\circ.$$

9. The sides of a triangle are 2, $\sqrt{6}+1$, $1+\sqrt{3}$. Calculate the angles.

10. The sides of a triangle angle are 16, 20, 33. Calculate the area.

11. If $A = 30^\circ$, $b = 12$, $a = 5$, is there any ambiguity?

12. If O be the middle point of the side BC of a triangle, show that $\cot CBA - \cot BCA = 2\cot COA$.

13. In a right-angled triangle, C being the right angle, show that

$$\cot\frac{A}{2} = \frac{b+c}{a}.$$

14. If two circles, whose radii are a and b, touch externally, and if θ be the angle contained in the common tangents to these circles, show that

$$\sin\theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

15. $15^\circ = 45^\circ - 30^\circ$; hence find all the trigonometrical ratios of 15°.

16. Find all the trigonometrical ratios of 105°.

17. The sides of a triangle are 13, 14, 15: find the radii of the inscribed and escribed circles. *Ans.* 4, $10\frac{1}{2}$, 12, 14.

18. Show that the sum of the cotangents of the halves of the three angles of a triangle is equal their product.

19. Show that one value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$.

20. Show that one value of $2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{5}{12}$.

21. Show that one value of $4\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{120}{119}$.

22. Show that one value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$.

23. Prove geometrically that $\frac{\pi}{4} = \tan^{-1} n + \tan^{-1}\left(\frac{1-n}{1+n}\right)$.

24. Solve the equation $\sin\theta + \cos\theta = \sqrt{2}$.

25. Solve the equation $\sqrt{3}\sin\theta + \cos\theta = \sqrt{2}$.

26. In any triangle show that

$$(a+b)\cos C + (b+c)\cos A + (c+a)\cos B = a+b+c.$$

27. In any triangle show that

$$1 - \tan\frac{A}{2} \,.\, \tan\frac{B}{2} = \frac{2c}{a+b+c}.$$

28. Show that the area of a triangle $= \frac{1}{4}(a^2\sin 2B + b^2\sin 2A)$.

29. Show that $a\cot A + b\cot B + \cot C =$ the sum of the diameters of the inscribed and circumscribed circles of any triangle.

30. Show that $\frac{a\cos A + b\cos B + c\cos C}{2\sin A \,.\, \sin B \,.\, \sin C} =$ the diameter of the circle circumscribing any triangle.

31. In any triangle show that

$$\frac{\tan A}{\tan B} = \frac{a^2+c^2-b^2}{b^2+c^2-a^2}.$$

32. Solve the equation $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$.

33. Solve the equation $\sin^{-1}(2x) + \sin^{-1}(3x) = \frac{\pi}{2}$.

34. What are the greatest and least values of sine?

35. What are the greatest and least values of versine?

36. What are the greatest and least values of tangent?

37. Prove that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$.

38. If $$A + B + C = 180^\circ,$$

prove $$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \,.\, \cos \frac{B}{2} \,.\, \cos \frac{C}{2}.$$

39. Prove $$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A+B}{2}.$$

40. Show that $\cot A - \tan A = 2 \cot 2A.$

41. Show that

$$\tan (A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \,.\, \tan B \,.\, \tan C}{1 - \tan A \,.\, \tan B - \tan A \,.\, \tan C - \tan B \tan C}.$$

42. Show that the angle whose circular measure is $\frac{3}{4}$ (nearly) $= 43^\circ$.

43. Find x from the equation $\tan x + \cot x = 2$.

44. Eliminate c between the equations

$$a^2 = b^2 + c^2 - 2bc \cos A, \quad \text{and} \quad b^2 = a^2 + c^2 - 2ac \cos B,$$

and reduce the result to the simplest form.

45. Prove $$\tan A = \frac{a \sin C}{b - a \cos C}.$$

46. If $\tan A = \frac{1}{7}$, $\tan B = \frac{2}{11}$, find $\tan (2A - B)$. *Ans.* $\frac{1}{2}$.

47. What angle is represented by the following expression?

$$\tan^{-1} \frac{1}{\sqrt{3}} + \tan^{-1}(1). \qquad \textit{Ans. } \frac{5\pi}{12}.$$

48. Given $\tan \frac{A}{2} = 2 - \sqrt{3}$, find $\sin A$. *Ans.* $\frac{1}{2}$.

49. If $\sin (r + \phi) = \mu \sin r$. Show that the circular measure of ϕ is approximately equal to $(\mu - 1) \tan r$, when ϕ is very small.

50. If $A = 60^\circ$, $a = \sqrt{3}$, $b = \sqrt{2}$, prove that

area of the $$\Delta = \sqrt{3}\left(\frac{1 + \sqrt{3}}{2}\right).$$

THE END.

DUBLIN UNIVERSITY PRESS SERIES.

THE

PARABOLA, ELLIPSE, AND HYPERBOLA,

TREATED GEOMETRICALLY.

BY

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From "THE ATHENÆUM," *April* 10, 1880.

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